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Crossover from percolation to self-organized criticality

Barbara Drossel, Siegfried Clar, and Franz Schwabl
Institut für Theoretische Physik, Physik-Department der Technischen Universität München,
James-Franck-Strasse, D-85747 Garching, Germany

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We include immunity against fire into the self-organized critical forest-fire model. When the immunity assumes a critical value, clusters of burnt trees are identical to percolation clusters of random bond percolation. As long as the immunity is below its critical value, the asymptotic critical exponents are those of the original self-organized critical model, i.e., the system performs a crossover from percolation to self-organized criticality. We present a scaling theory and computer simulation results.

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Several extended nonequilibrium systems show scaling behavior over a wide range of parameter values and independently of the initial conditions. They are called *self-organized critical* (SOC) [1] and might explain the ubiquity of fractal structures in nature. By analogy to equilibrium critical phenomena, the question arises if SOC phenomena are universal, i.e., if the critical behavior depends only on a few properties of the model such as dimension, conservation laws, or number of components. So far, examples for both universal and nonuniversal behavior have been found. While the critical behavior of the sandpile model [1] seems to be robust with respect to various changes, the critical exponents of an earthquake model [2] change continuously as a function of a parameter which characterizes the degree of conservation. In equilibrium critical phenomena, such a continuous change of critical exponents occurs only in exceptional cases, e.g., the eight vertex model. Instead, critical behavior is characterized by a crossover between two fixed points, when certain parameters are changed. This crossover is described by scaling functions and a crossover exponent ϕ. In this paper, we report a crossover phenomenon in SOC systems by including the immunity g in the SOC forest-fire model [3]. g is the probability that fire cannot pass from one tree to a neighboring tree. When the immunity assumes a critical value g_c, the forest becomes very dense, and clusters that are burnt by a lightning stroke are identical to percolation clusters. As long as the immunity is below its critical value, the asymptotic critical exponents of the model are those of the original SOC forest-fire model, i.e., the system performs a crossover from percolation to SOC.

The forest-fire model is defined on a d-dimensional hypercubic lattice with L^d sites. Each site is either occupied by a tree, a burning tree, or it is empty. The state of the system is parallel updated according to the following rules.

(i) Burning tree→empty site.

(ii) Tree→burning tree with probability $1-g^n$ if $n\geqslant 1$ nearest neighbors are burning.

(iii) Tree→burning tree with probability f if no nearest neighbor is burning.

(iv) Empty site→tree with probability p.

Starting with arbitrary initial conditions, after some time the system approaches a steady state, the properties of which depend on the parameter values but not on the initial state. We always chose the lattice size so large that the steady state is also independent of the boundary conditions. Let ρ_t be the density of trees, ρ_e the density of empty sites, and ρ_f the density of burning trees in the steady state. Large forests and therefore large fires occur only if the lightning probability f is much smaller than the tree growth probability p. If additionally large and small fires look alike, the system becomes scale invariant. Small fires live only for a few time steps and are extinguished before new trees grow in their neighborhood. Scale invariance therefore can be observed only if tree growth is so slow that large fires also are extinguished before new trees grow at the edge of the burning forests. We conclude that SOC behavior occurs in the forest-fire model if the following conditions are satisfied: Lightning occurs seldom compared to tree growth, and tree growth is much slower than the lifetime of a fire, i.e.,

$$f \ll p \ll (f/p)^{\nu'} \tag{1}$$

with an appropriate exponent ν' [3]. Equation (1) represents a double separation of time scales. Since in the steady state the mean number of burning trees equals the mean number of growing trees, the mean number of trees destroyed by a lightning stroke is [3]

$$\bar{s} = \frac{\rho_e}{\rho_t}\frac{p}{f} \simeq \frac{1-\rho_t}{\rho_t}\frac{p}{f}. \tag{2}$$

In the last step, we have neglected the fire density ρ_f which vanishes in the limit of perfect time scale separation. In $d\geqslant 2$ dimensions and for values of the immunity below its critical value, the critical forest density $\rho_t^c = \lim_{f/p\to 0}\rho_t$ is smaller than 1, and Eq. (2) represents a power law $\bar{s} \propto (f/p)^{-\gamma}$ with $\gamma=1$, indicating a critical point at $f/p=0$. Close to this critical point, i.e., if $f \ll p$, there is scaling over many orders of magnitude.

Several critical exponents characterizing this scaling behavior have been defined [3–5]. The forest density satisfies for small f/p a power law

$$\rho_t^c - \rho_t \propto (f/p)^{1/\delta}, \tag{3}$$

as first stated in [4] and [6].

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Let s be the number of trees destroyed by a fire. The size distribution of fires is a power law [3]

$$sn(s) \propto s^{1-\tau} C(s/s_{\max}) \tag{4}$$

with

$$s_{\max} \propto (f/p)^{-\lambda}. \tag{5}$$

Inserting Eq. (5) and Eq. (4) into Eq. (2), we obtain the scaling relation $\lambda = 1/(3-\tau)$, which is valid as long as $\rho_t^c < 1$. In general, if we allow also the case $\rho_t^c = 1$, as relevant for $g = g_c$, $\bar{s}$ is given by a power law $\bar{s} \propto (f/p)^{-\gamma}$, with a value for γ which may be different from 1, and we have the more general scaling relation

$$\lambda = \gamma/(3-\tau)\ . \tag{6}$$

We also define the radius $R(s)$ of a forest cluster that has just been burnt down. Its fractal dimension μ is given by

$$R(s) \propto s^{1/\mu}. \tag{7}$$

The correlation length ξ is defined by

$$\xi^2 = \frac{2\sum_{s=1}^{\infty} R^2(s) s^2 n(s)}{\sum_{s=1}^{\infty} s^2 n(s)}\ . \tag{8}$$

Together with Eqs. (4), (5), and (7), this gives

$$\xi \propto (f/p)^{-\nu} \quad \text{with} \quad \nu = \lambda/\mu\ . \tag{9}$$

In the SOC forest-fire model without immunity, these exponents have already been determined by computer simulations [4–7]. In $d=2$ dimensions, they are [5]

$$\begin{gathered} \tau_{\mathrm{SOC}} = 2.14(3), \quad \lambda_{\mathrm{SOC}} = 1.15(3), \quad \mu_{\mathrm{SOC}} = 1.96(1), \\ \nu_{\mathrm{SOC}} = 0.58, \quad 1/\delta_{\mathrm{SOC}} = 0.48(2). \end{gathered} \tag{10}$$

In $d=1$ dimension, their values have been derived analytically [8]. In [5], it is also shown that the critical exponents in two dimensions do not change when the lattice symmetry is changed or next-nearest neighbor interaction is included.

We now consider the SOC forest-fire model with nonvanishing immunity, i.e., for $g > 0$. Immunity was first introduced in [9] into a version of the forest-fire model [10] which is not SOC [11,12]. This model shows a percolation-like phase transition when the immunity approaches a critical value g_c. Now the immunity is included into the SOC forest-fire model. It is defined differently than before: While in [9] the immunity is a property of trees, in the present paper it is a property of bonds between neighboring trees [see rule (ii) above]. This has two advantages: The simulation program is less complicated, and the critical immunity is known to be exactly $g_c = 0.5$ which is just 1 minus the percolation threshold for bond percolation.

When the immunity is different from 0, not all trees that are neighbors of a burning tree catch fire, and consequently the fire no longer burns forest clusters but clusters of trees that are connected by nonimmune bonds. With increasing immunity, the forest density increases, since fewer trees are burnt. At the critical immunity g_c, the critical forest density is $\rho_t^c = 1$. Then we have the following situation: The forest is completely dense in the limit $f/p \rightarrow 0$, and clusters that are destroyed by fire are percolation clusters of bond percolation. Consequently, the exponents τ and μ are given by percolation theory

$$\tau(g_c) \equiv \tau_c = \tau_{\mathrm{perc}} = 187/91 \simeq 2.05 \tag{11}$$

and

$$\mu(g_c) \equiv \mu_c = \mu_{\mathrm{perc}} = 91/48 \simeq 1.90. \tag{12}$$

Bonds that are immune during one time step might be nonimmune during the next one, and consequently lightning strokes at the same site burn down different clusters at different times. When f/p is finite, there is a cutoff in cluster size, since large fires are stopped by empty sites that have been left from earlier fires. The mean forest density is no longer 1. We determined the critical exponents λ, δ, and ν at $g = g_c$ by computer simulations in $d=2$ dimensions and obtained

$$\lambda_c = 0.92(3), \quad 1/\delta_c = 0.15(1), \quad \nu_c = 0.484(2). \tag{13}$$

The scaling relations Eq. (6) and Eq. (9) still hold at $g = g_c$ and are confirmed by our simulations. Using Eqs. (2) and (3), we obtain

$$\bar{s} \propto (f/p)^{-\gamma_c} \quad \text{with} \quad \gamma_c = 1 - 1/\delta_c\ . \tag{14}$$

Our simulations yield $\gamma_c = 0.84(2)$, in agreement with Eq. (14).

When the immunity is just below its critical value $[(g_c - g) \ll 1]$, the situation becomes more complicated. On small length scales, a system close to the percolation threshold cannot be distinguished from a system exactly at the percolation threshold. On large length scales, however, the difference can be seen. If the initial state was a completely dense forest, a finite fraction of all trees were connected by nonimmune bonds for $g < g_c$. This infinite cluster would soon be destroyed by lightning and would never occur again due to Eq. (2). Consequently, the critical forest density is $\rho_t^c < 1$ for $g < g_c$. Imagine for a moment that bonds are permanently immune or nonimmune [in reality, rule (ii) says that they might be immune during one time step and nonimmune during the next one]. Then the dynamics on the sites which initially belonged to the infinite cluster are completely decoupled from dynamics on the remaining sites. We can consider the sites which belonged to the infinite cluster as an independent subsystem where no immunity exists. They form a two-dimensional lattice with another symmetry than the original hypercubic one. Since we already know from earlier simulations that the exponents of the SOC forest-fire model do not change when lattice symmetry is changed, we expect that the critical exponents on our subsystem are just the ones which we obtained for $g=0$ and which have been assigned an index "SOC." The remaining sites are connected by nonimmune bonds to form finite clusters of bond percolation. On each of these clusters, the dynamics are independent of the other clusters and of the infinite subsystem. The largest of these clusters has a radius of the order of the percolation correlation length $\xi_{\text{perc}} \propto (g_c-g)^{-\nu_{\text{perc}}}$. When f/p becomes very small, these finite clusters are very rarely struck by lightning, and consequently the mean tree density on these clusters approaches the value 1 in the limit $f/p \rightarrow 0$. Each time lightning strikes such a cluster, there is a fire which has the size and the fractal dimension of a finite percolation cluster. As long as f/p is so small that the correlation length $\xi \propto (f/p)^{-\nu_c}$ is much larger than the percolation correlation length ξ_{perc}, all large fires occur on the infinite subsystem, and the exponents ν, δ, and λ are those of the subsystem, i.e., the SOC ones. The exponents τ and μ of the large fires are by the same reasoning also the SOC ones. But for fires with a radius smaller than the percolation correlation length, τ and μ assume their percolation values since the infinite subsystem contains only a small portion of all sites and consequently most of the small fires occur on the finite clusters. When f/p becomes so large that the forest clusters on our infinite subsystem are no more larger than the percolation correlation length, our system is dominated by the dynamics on the finite clusters and becomes indistinguishable from a system at $g=g_c$. The exponents ν, δ, and λ then are identical to those at $g=g_c$, and the exponents τ and μ are those of percolation theory on all length scales up to the correlation length. So far, we considered the case of small g_c-g. When the immunity is far away from its critical value, the infinite subsystem contains a large portion of all sites, and the percolation-dominated behavior cannot occur anymore.

Unfortunately, all these considerations are based on the assumption that bonds are permanently immune or nonimmune, which in reality is not the case. Consequently, there exists no subsystem which is decoupled from the rest of the system. Nevertheless, the main conclusions should remain valid: On length scales smaller than the percolation correlation length, the system cannot be distinguished from a system at $g=g_c$. When f/p becomes very small, there are fires which spread farther than the percolation correlation length. These fires are stopped by empty sites that were created by earlier fires. This is again the same mechanism as in the limit $g=0$ or on the infinite subsystem in the case of fixed immune bonds: fires that would spread indefinitely if there were no empty sites are stopped by empty sites. We conclude that these large fires lead again to the critical exponents λ_{SOC}, ν_{SOC}, and δ_{SOC}.

Led by these considerations, we make the following scaling ansatz for the correlation length:

$$\xi=(f/p)^{-\nu_c}F\left(\frac{g_c-g}{(f/p)^{\phi}}\right). \tag{15}$$

It is plausible that the crossover from percolationlike to SOC behavior takes place when f/p becomes so small that the correlation length exceeds the percolation correlation length, which suggests that the crossover exponent ϕ is

$$\phi=\nu_c/\nu_{\text{perc}}. \tag{16}$$

The scaling function $F(x)$ is constant for small x and is $\propto x^{(\nu_{\text{SOC}}-\nu_c)/\phi}$ for large x. Analogous scaling laws hold for $s_{\max}$ and $\rho_t^c-\rho_t$. We already mentioned above that the critical forest density is $\rho_t^c=1$ at g_c. We therefore expect an additional power law

$$1-\rho_t^c(g)\propto(g_c-g)^y. \tag{17}$$

The exponent y is obtained from the scaling ansatz

$$1-\rho_t=(f/p)^{1/\delta_c}G\left(\frac{g_c-g}{(f/p)^{\phi}}\right). \tag{18}$$

In the limit $f/p \rightarrow 0$, the forest density becomes independent of f/p and assumes a value $\rho_t^c \neq 1$. Therefore $G(x) \propto x^{1/\phi\delta_c}$ for large x, yielding

$$y=\nu_{\text{perc}}/\nu_c\delta_c. \tag{19}$$

Our simulations confirm all these results. They were performed using the same method as in [5,6] for lattices of up to 8192^2 sites and values of f/p down to 10^{-6}. The immunity varied between $(g_c-g)=0.02$ and $(g_c-g)=0.001$. The range of f/p is limited by finite-size effects for small f/p and noncritical behavior for large f/p, the range of g was selected from the condition that at least part of the crossover region is covered. Figure 1 shows the scaling function for the correlation length $F(x)$ for different values of g_c-g. The scaling ansatz Eq. (15) is well confirmed since all curves

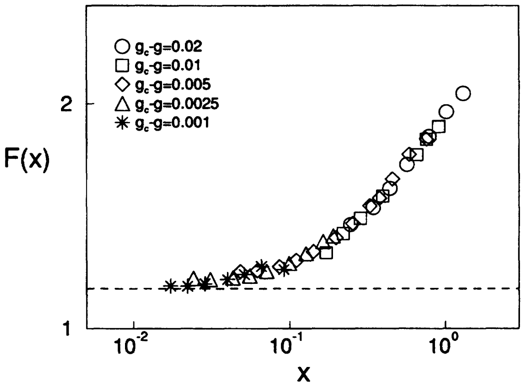

FIG. 1. Crossover scaling function $F(x)$ for the correlation length for different values of the immunity. The dashed line represents $F(0)$ as obtained at $g=g_c$.

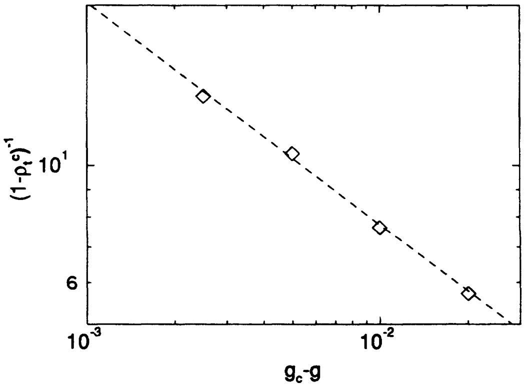

FIG. 2. The critical forest density as a function of the immunity.

coincide. The dashed line represents $F(0)$ as obtained from the simulations at g_c. We also checked the scaling relation Eq. (19). Figure 2 shows the critical forest density as a function of g_c. We obtain $y=0.43(2)$, in agreement with Eq. (19) (remember that $\nu_{\text{perc}}=4/3$). Since the difference between τ_{SOC} and τ_{perc}, as well as between μ_{SOC} and μ_{perc}, is very small, the crossover behavior of $n(s)$ and $R(s)$ could not be evaluated.

In [5], we also defined exponents ν', μ', and α describing the temporal behavior of the fires. The crossover in μ' and ν', which enters the condition for time scale separation Eq. (1), is analogous to the crossover in μ and ν [13]. The change in α is too small for any crossover to be observable.

To conclude, we have shown by analytic arguments and by computer simulations that the forest-fire model performs a crossover from percolation to SOC when the immunity is close to its critical value. This crossover is characterized by scaling functions which are defined in the same way as in crossover phenomena at equilibrium phase transitions.

Although all simulations were performed in $d=2$ dimensions, we expect that this crossover behavior can also be observed in higher dimensions. In $d=1$, the critical immunity is $g_c=0$, and no crossover can take place. For $d\geqslant 6$, simulations suggest that the critical exponents assume their mean-field values which are identical to those of percolation [5,7]. Consequently there is no crossover in $d\geqslant 6$ dimensions.

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